FINAL: ALGEBRA III

Date: **2nd Jan 2019**
The total points is **100**.

A **ring** would mean a **commutative ring with identity**.

(1) (5+15=20 points) When is a ring called noetherian? Let A be the localization of $\mathbb{Z}[x, y]$ with respect to a multiplicative subset S. Show that A is noetherian and UFD.

(2) (4+8+8=20 points) Let R be a ring and M be an R-module. When is an element of $x \in M$ called a torsion element? Give a counter example to disprove the following statements.
   (a) The set $Tor(M)$ consisting of torsion elements of M is an R-submodule of M.
   (b) Every torsion free $\mathbb{Z}$-module is free.

(3) (5+15=20 points) Define local ring. Let R be a local ring and a Principal Ideal Domain with field of fractions K. Let $a \in R$ be a nonzero nonunit. Show that the subring $R[a^{-1}]$ of K is the field K itself.

(4) (20 points) Let R be a ring, M be an R-module and N be an R-submodule of M. Show that if N is noetherian and M/N is noetherian then M is noetherian.

(5) (5+15=20 points) State structure theorem for finitely generated modules over a PID. Compute the rational canonical and the Jordan canonical forms of the following matrix. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$